Limit of Sequences

1. Show that y^n and ny^n both tend to zero as n tends to infinity if $0 < y < 1$.
Show that $(1 + a)^n < (1 - na)^{-1}$ is n is a positive integer, $a > 0$ and $na < 1$,
and then prove that, if $0 < x < 1$, $(1 + x^n)^n$ tends to 1 as n tends to infinity.

2. The numbers of a sequence $u_0, u_1, u_2, \dots$ are given by the relation

$$u_n - (k + k^{-1})u_{n-1} + u_{n-2} = 0 \qquad (n \ge 2 , k \ne 1).$$

Prove that if $u_0 = 1$ and $k > 1$, then the unique value of u such that u_n tends to a limit as $n \to \infty$ is $\dfrac{1}{k}$.

3. If $a > 0$ and $0 < x_1 < b$, show that the sequence $\{x_n\}$ defined by the formula

$$x_n = \sqrt{\frac{ab^2 + x_n^{\ 2}}{a+1}}$$ is an increasing bounded sequence and find its limit.

4. The sequence $\{x_n\}$ is defined by the relation $x_{n+1} = x_n^{\ 2} - 2x_n + 2$ $(n > 1)$.
Investigate the behaviour of x_n in the cases
(a) $1 < x_1 < 2$ **(b)** $x_1 = 2$ **(c)** $x_1 > 2$.

5. Given that $a_0 = a_1 \sin^2 \phi = 2 \cos \phi$ and that $a_n - 2a_{n+1} + a_{n+2} \sin^2 \phi = 0$.
Find an expression for a_1 and show that $\displaystyle\sum_{r=1}^{n} a_r = \left(\frac{1}{1-\cos\phi}\right)^n - \left(\frac{1}{1+\cos\phi}\right)^n$.

6. A sequence is defined by the formula $u_{n+1} = \dfrac{6u_n^{\ 2}+6}{u_n^{\ 2}+11}$, $n = 0, 1, 2, \dots$ and by the value of the initial term u_0 . Prove that if u_n converges to a limit a , then a is either 1, 2, 3.
Show that, if $u_0 > 3$, then **(i)** $3 < u_{n+1} < u_n$ **(ii)** $\dfrac{u_{n+1}-3}{u_n-3} < \dfrac{9}{10}$
and show that, when $u_0 > 3$, then $u_n \to 3$ as $n \to \infty$.

7. If $\{x_n\}$ is defined by $x_0 = x$, $x_{n+1} = \dfrac{x_n}{1+\sqrt{1+x_n^{\ 2}}}$ $(n = 0, 1, 2, \dots)$ and a_n , b_n are defined by $a_n = 2^n x_n$, $b_n = \dfrac{2^n x_n}{\sqrt{1+x_n^{\ 2}}}$. Prove that $\{a_n\} , \{b_n\}$ both converge to the same limit.

8. If $0 < a_1 < 3$ and $a_{n+1} = \dfrac{12}{1+a_n}$, show that the sequences $\{a_{2n+1}\}$ and $\{a_{2n}\}$ are respectively increasing and decreasing monotonically. Prove that the sequence a_n converges to the limit 3 .

9. Positive numbers $u_1, u_2 , \dots$ are defined by $u_1 = 3$, $u_{n+1} = \sqrt{u_n + 5}$, $(n = 0, 1, 2, \dots)$
Prove that as $n \to \infty$, u_n tends to a limit a . Prove also that $0 < u_{2n+1} - a < \dfrac{1}{30^n}(u_1 - a)$.

10. Points $P_0(x_0, y_0)$, $P_1(x_1, y_1)$, …, $P_n(x_n, y_n)$ are connected by the following relations :

$x_{n+1} = 2x_n + 3y_n$, $y_{n+1} = x_n + 2y_n$, $x_0 = 1$, $y_0 = 0$.

(i) Find a if the items of the sequence $\{z_n\} = \{x_n + ay_n\}$ forms a geometric sequence.

(ii) Find the limit of the gradient of the straight line OP_n when $n \to \infty$.

(You may assume that the limit exists.)

11. The sequence $a_1, a_2, a_3, \ldots$ is defined as $a_1 = 3$, $a_{n+1} = \dfrac{a_n^{\,2} + 5}{2a_n}$, $n > 0$.

Prove that $0 < a_{n+1} - \sqrt{5} < \dfrac{\left(3-\sqrt{5}\right)^{2^n}}{\left(2\sqrt{5}\right)^{2^n-1}} < 6 \times \left(\dfrac{2}{11}\right)^{2^n}$ and find $\lim\limits_{n\to\infty} a_n$.

12. A sequence $\{r_n\}$ is defined as follows, $r_0 > 0$, $r_{n+1} + \dfrac{1}{r_n} = 2A$, $A > 0$.

Show that the condition that is necessary for the convergence of r_n is that $A \ge 1$. Is the condition that $A \ge 1$ sufficient for the convergence of r_n ?

13. Show that $\left\{\sin \dfrac{n\pi}{5}\right\}$ is divergent.

14. Show that $\left\{1 + \dfrac{1}{3} + \ldots + \dfrac{1}{2n-1}\right\}$ is divergent.

15. Let $\{a_n\}$ be a sequence of positive numbers. Define $b_n = \left(a_1 a_2 \ldots a_n\right)^{1/n}$ for $n > 0$.

If $\lim\limits_{n\to\infty} a_n = \alpha$, show that $\lim\limits_{n\to\infty} b_n = \alpha$. Hence show that if $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \alpha$, then $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \alpha$.

Show $\lim\limits_{n\to\infty} \dfrac{1}{n}(n!)^{1/n} = \lim\limits_{n\to\infty} n^{1/n} = 1$, if it is given that $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$.

16. If $x_1 = h$, $x_{n+1} = x_n^{\,2} + k$, where $0 < k < \dfrac{1}{4}$ and h lies between the roots a , b of $x^2 - x + k = 0$. Prove that $a < x_{n+1} < x_n < h$ and determine the limit of x_n .

17. Define a sequence of numbers $x_1, x_2, \ldots$ by $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$, $a \ge 0$.

Show that $\sqrt{a} < x_n < x_{n-1} < \ldots < x_0$, provided that x_0 is positive and $x_0 > \sqrt{a}$.

Find the limit of $\{x_n\}$. Does $\{x_n\}$ converge if $a < 0$?

18. Given two real numbers x_1 and x_2 . Let $x_n = \dfrac{x_{n-1} + x_{n-2}}{2}$, $n > 2$.

Find the limit of the sequence x_n as n tends to infinity.

19. If $0 < x < \dfrac{\pi}{2}$, using the fact that $\cos\dfrac{x}{2}\cos\dfrac{x}{2^2}\ldots\cos\dfrac{x}{2^n} = \dfrac{\sin x}{2^n \sin\left(\dfrac{x}{2^n}\right)}$, find $\lim\limits_{n\to\infty}\cos\dfrac{x}{2}\cos\dfrac{x}{2^2}\ldots\cos\dfrac{x}{2^n}$.

Let $\mu > \lambda > 0$, the sequences $\{x_n\}$, $\{y_n\}$ are defined by

$x_1 = \lambda$, $x_2 = \dfrac{x_1 + y_1}{2}, \ldots, x_n = \dfrac{x_{n-1} + y_{n-1}}{2}$, $y_1 = \mu$, $y_2 = \sqrt{x_2 y_1}$,..., $y_n = \sqrt{x_n y_{n-1}}$,

putting $\lambda = \mu \cos x$, show that the two sequences has the same limit $\dfrac{\sqrt{\mu^2 - \lambda^2}}{\cos^{-1}\left(\dfrac{\lambda}{\mu}\right)}$.

20. Prove that n is a positive integer, then $\left(\sqrt{3}+1\right)^n = a_n\sqrt{3} + b_n$ for unique integer a_n , b_n .

Further, prove that

(a) $a_{n+2} = 2\left(a_{n+1} + a_n\right)$, $b_{n+2} = 2\left(b_{n+1} + b_n\right)$;

(b) $\left(\sqrt{3}-1\right)^n = (-1)^{n-1}\left(a_n\sqrt{3} - b_n\right)$;

(c) $3{a_n}^2 - {b_n}^2 = (-1)^{n-1}2^n$;

(d) $\lim\limits_{n\to\infty}\dfrac{b_n}{a_n} = \sqrt{3}$.

21. For any sequence $\{a_n\}$ of real number, it is known that if $a_n \neq 0$, $\lim\limits_{n\to\infty} a_n = 0$,then $\lim\limits_{n\to\infty}\dfrac{a_n}{\sin a_n} = 1$

and $\lim\limits_{n\to\infty} a_{2n} = 0$.

(a) Let $P_n = \cos\dfrac{x}{2}\cos\dfrac{x}{2^2}\ldots\cos\dfrac{x}{2^n}$, where $0 < x < \dfrac{\pi}{2}$. Evaluate $\lim\limits_{x\to\pi/2}\left(\lim\limits_{n\to\infty} P_n\right)$.

(b) Show that there exists no real number x such that $\cos nx$ tends to zero as n tends to infinity.

(c) Consider the differences of $\sin(n+2)x$ and $\sin(n+1)x$, hence prove that $\lim\limits_{n\to\infty}\sin nx = 0$ if and only if $x = m\pi$ for some integer m .

22. Let x be a non-zero real number greater than -1 . Prove by induction that for any positive integer n greater than 1, $(1 + x)^n > 1 + nx$.

Let t be any fixed positive number. Consider the sequence $a_n = \sqrt[n]{t}$

(a) Let $t > 1$. Show that $\sqrt[n]{t} > 1$.

(b) Putting $\sqrt[n]{t} = 1 + x_n$ and using **(a)** or otherwise, show that $1 < \sqrt[n]{t} < 1 + \dfrac{t-1}{n}$ for $n > 2$.

Hence show that for $t > 1$, $\lim\limits_{n\to\infty} a_n = 1$.